

# The Douglas–Rachford algorithm in the affine-convex case

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## Abstract

The Douglas–Rachford algorithm is a simple yet effective method for solving convex feasibility problems. However, if the underlying constraints are inconsistent, then the convergence theory is incomplete. We provide convergence results when one constraint is an affine subspace. As a consequence, we extend a result by Spingarn from halfspaces to general closed convex sets admitting least-squares solutions.

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## 1 Introduction

We shall assume throughout this paper that  $X$  is a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and induced norm  $\|\cdot\|$ , and that

$A$  and  $B$  are nonempty closed convex (not necessarily intersecting) subsets of  $X$ . (1)

Consider the problem of finding a best approximation pair relative to  $A$  and  $B$  (see [3], [9]), that is to

find  $(a, b) \in A \times B$  such that  $\|a - b\| = \inf \|A - B\|$ . (2)

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Recall that the Douglas–Rachford splitting operator [8] for the ordered pair of sets  $(A, B)$  is defined by

$$T = T_{(A,B)} := \frac{1}{2}(\text{Id} + R_B R_A) = \text{Id} - P_A + P_B R_A, \quad (3)$$

where  $P_A$  is the projector onto  $A$  and  $R_A := 2P_A - \text{Id}$  is the reflector onto  $A$ . Let  $x \in X$ . In the consistent case, when  $Z_{A,B} := A \cap B \neq \emptyset$ , the “governing sequence”  $(T^n x)_{n \in \mathbb{N}}$  generated by iterating the Douglas–Rachford operator converges weakly to a fixed point<sup>1</sup> of  $T$  (see [8]), and the “shadow sequence”  $(P_A T^n x)_{n \in \mathbb{N}}$  converges weakly to a point in  $A \cap B$  (see [12] or [2, Theorem 25.6]). For further information on the Douglas–Rachford algorithm (DRA), see also [8] and [6].

In [3], the authors showed that in the inconsistent case, when  $A \cap B = \emptyset$ ,  $(P_A T^n x)_{n \in \mathbb{N}}$  remains bounded with the weak cluster points of  $(P_A T^n x, P_B P_A T^n x)_{n \in \mathbb{N}}$  being best approximation pairs relative to  $A$  and  $B$  whenever  $g := P_{B-A} 0 \in B - A$ . The goal of this paper is to study the case when  $A \cap B$  is possibly empty in the setting that one of the sets  $A$  and  $B$  is a closed affine subspace of  $X$ . Our results show that the shadow sequence will always converge to a best approximation solution in  $A \cap (B - g)$ . As a consequence we obtain a far-reaching refinement of Spingarn’s splitting method introduced in [11].

## 2 Main results

We start with the following key lemma, which is well known when  $A = X$ .

**Lemma 2.1.** *Let  $A$  be a closed linear subspace of  $X$ , let  $C$  be a nonempty closed convex subset of  $A$ , and let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . Suppose that  $(x_n)_{n \in \mathbb{N}}$  is Fejér monotone with respect to  $C$ , i.e.,  $(\forall n \in \mathbb{N}) (\forall c \in C) \|x_{n+1} - c\| \leq \|x_n - c\|$ , and that all its weak cluster points of  $(P_A x_n)_{n \in \mathbb{N}}$  lie in  $C$ . Then  $(P_A x_n)_{n \in \mathbb{N}}$  converges weakly to some point in  $C$ .*

*Proof.* Since  $(x_n)_{n \in \mathbb{N}}$  is bounded (by e.g., [2, Proposition 5.4(i)]) and  $P_A$  is (firmly) nonexpansive we learn that  $(P_A x_n)_{n \in \mathbb{N}}$  is bounded and by assumption, its weak cluster points lie in  $C \subseteq A$ . Now let  $c_1$  and  $c_2$  be in  $C$ . On the one hand the Fejér monotonicity of  $(x_n)_{n \in \mathbb{N}}$  implies the convergence of the sequences  $(\|x_n - c_1\|^2)_{n \in \mathbb{N}}$  and  $(\|x_n - c_2\|^2)_{n \in \mathbb{N}}$  by e.g., [2, Proposition 5.4(ii)]. On the other hand, expanding and simplifying yield  $\|x_n - c_1\|^2 - \|x_n - c_2\|^2 = \|x_n\|^2 + \|c_1\|^2 - 2\langle x_n, c_1 \rangle - \|x_n\|^2 - \|c_2\|^2 + 2\langle x_n, c_2 \rangle = \|c_1\|^2 - 2\langle x_n, c_1 - c_2 \rangle - \|c_2\|^2$ , which in turn implies that  $(\langle x_n, c_1 - c_2 \rangle)_{n \in \mathbb{N}}$  converges. Since  $c_1 \in A$  and  $c_2 \in A$  we have

$$\langle x_n, c_1 - c_2 \rangle = \langle x_n, P_A c_1 - P_A c_2 \rangle = \langle x_n, P_A (c_1 - c_2) \rangle = \langle P_A x_n, c_1 - c_2 \rangle. \quad (4)$$

Now assume that  $(P_A x_{k_n})_{n \in \mathbb{N}}$  and  $(P_A x_{l_n})_{n \in \mathbb{N}}$  are subsequences of  $(P_A x_n)_{n \in \mathbb{N}}$  such that  $P_A x_{k_n} \rightharpoonup c_1$  and  $P_A x_{l_n} \rightharpoonup c_2$ . By the uniqueness of the limit in (4) we conclude that  $\langle c_1, c_1 - c_2 \rangle = \langle c_2, c_1 - c_2 \rangle$  or equivalently  $\|c_1 - c_2\|^2 = 0$ , hence  $(P_A x_n)_{n \in \mathbb{N}}$  has a unique weak cluster point which completes the proof.  $\blacksquare$

<sup>1</sup> $\text{Fix } T = \{x \in X \mid x = Tx\}$  is the set of fixed points of  $T$ .

From now on we work under the assumption that

$$g = g_{(A,B)} := P_{\overline{B-A}}0 \in B - A. \quad (5)$$

In view of (5) we have

$$E = E_{(A,B)} := A \cap (B - g) \neq \emptyset \quad \text{and} \quad F = F_{(A,B)} := (A + g) \cap B \neq \emptyset. \quad (6)$$

For sufficient conditions on when  $g \in B - A$  (or equivalently the sets  $E$  and  $F$  are nonempty) we refer the reader to [1, Facts 5.1].

**Lemma 2.2.** *Let  $x \in X$ . Then the following hold:*

- (i) *If  $C \in \{A, B\}$  is a closed affine subspace of  $X$ , then  $g \in (C - C)^\perp$ .*
- (ii) *The sequence  $(T^n x - ng)_{n \in \mathbb{N}}$  is Fejér monotone with respect to  $E$ .*
- (iii) *The sequence  $(P_A T^n x)_{n \in \mathbb{N}}$  is bounded and its weak cluster points lie in  $E$ .*
- (iv) *If  $B$  is a closed affine subspace, then  $P_B T^n x - P_A T^n x \rightarrow g$ , the sequence  $(P_B T^n x)_{n \in \mathbb{N}}$  is bounded and all weak cluster points lie in  $F$ .*
- (v) *If  $E = \{\bar{x}\}$  and hence  $F = \{\bar{x} + g\}$ , then  $P_A T^n x \rightharpoonup \bar{x}$  and  $P_B T^n x \rightharpoonup \bar{x} + g$ .*

*Proof.* (i): See [3, Corollary 2.7 and Remark 2.8(ii)]. (ii): It follows<sup>2</sup> from [3, Theorem 3.5] that  $E + N_{\overline{A-B}}(-g) \subseteq \text{Fix}(-g + T) := \{x \in X \mid x = -g + Tx\} \subseteq -g + E + N_{\overline{A-B}}(-g)$ . Consequently,  $E \subseteq \text{Fix}(-g + T)$ . Moreover, [3, Remark 3.15] implies that the sequence  $(T^n x - ng)_{n \in \mathbb{N}}$  is Fejér monotone with respect to  $\text{Fix}(-g + T)$ . (iii): See [3, Theorem 3.13(iii)(b)]. (iv): See [3, Theorem 3.17]. (v): This follows from (iii) and (iv). ■

We are now ready for our main results.

**Theorem 2.3 (convergence of DRA when  $A$  is a closed affine subspace).** *Suppose that  $A$  is a closed affine subspace of  $X$ , and let  $x \in X$ . Then the following hold:*

- (i) *The shadow sequence  $(P_A T^n x)_{n \in \mathbb{N}}$  converges weakly to some point in  $E = A \cap (B - g)$ .*
- (ii) *No general conclusion can be drawn about the sequence  $(P_B T^n x)_{n \in \mathbb{N}}$ .*

*Proof.* (i): After translating the sets  $A$  and  $B$  by a vector, if necessary, we can and do assume that  $A$  is a closed linear subspace of  $X$ . Using Lemma 2.2(i) we learn that  $(\forall n \in \mathbb{N}) P_A T^n x = P_A(T^n x - ng)$ . Note that  $E = A \cap (B - g) \subseteq A$ . Now combine Lemma 2.2(ii)–(iii) and Lemma 2.1 with  $C = E$ , and  $(x_n)_{n \in \mathbb{N}}$  replaced by  $(T^n x - ng)_{n \in \mathbb{N}}$ . (ii): In fact,  $(P_B T^n x)_{n \in \mathbb{N}}$  can be unbounded (see Example 2.4) or bounded (e.g., when  $A = B = X$ ). ■

<sup>2</sup> We use  $N_C$  to denote the *normal cone* operator associated with a nonempty closed convex subset  $C$  of  $X$ .

**Example 2.4.** Suppose that  $X = \mathbb{R}^2$ , that  $A = \mathbb{R} \times \{0\}$  and that  $B = \text{epi}(|\cdot| + 1)$ . Then  $A \cap B = \emptyset$  and for the starting point  $x \in [-1, 1] \times \{0\}$  we have  $(\forall n \in \{1, 2, \dots\}) T^n x = (0, n) \in B$  and therefore  $\|P_B T^n x\| = \|T^n x\| = n \rightarrow \infty$ .

*Proof.* Let  $x = (\alpha, 0)$  with  $\alpha \in [-1, 1]$ . We proceed by induction. When  $n = 1$  we have  $T(\alpha, 0) = P_{A^\perp}(\alpha, 0) + P_B R_A(\alpha, 0) = P_B(\alpha, 0) = (0, 1)$ . Now suppose that for some  $(n \in \{1, 2, \dots\}) T^n x = (0, n)$ . Then  $T^{n+1}x = T(0, n) = P_{A^\perp}(0, n) + P_B R_A(0, n) = (0, n) + P_B(0, -n) = (0, n+1) \in B$ . ■

When  $B$  is an affine subspace, the convergence theory is even more satisfying:

**Theorem 2.5 (convergence of DRA when  $B$  is a closed affine subspace).** *Suppose that  $B$  is a closed affine subspace of  $X$ , and let  $x \in X$ . Then the following hold:*

- (i) *The shadow sequence  $(P_A T^n x)_{n \in \mathbb{N}}$  converges weakly to some point in  $E = A \cap (B - g)$ .*
- (ii) *The sequence  $(P_B T^n x)_{n \in \mathbb{N}}$  converges weakly to some point in  $F = (A + g) \cap B$ .*

*Proof.* (ii): Combine Theorem 2.3(i) and [5, Corollary 2.8(i)]. (i): Combine (ii) and Lemma 2.2(iv). ■

It is tempting to conjecture that Theorem 2.3(i) remains true when  $A$  is just convex and not necessarily a subspace. While this *statement* may be true<sup>3</sup>, the *proof* of Theorem 2.3(i) does *not* admit such an extension:

**Example 2.6.** Suppose that  $X = \mathbb{R}$ , that  $A = [1, 2]$  and that  $B = \{0\}$ . Then  $g = -1$  and  $E = \{1\}$ . Let  $x = 4$ . We have  $(T^n x)_{n \in \mathbb{N}} = (4, 2, 0, -1, -2, -3, \dots)$ ,  $P_A T^n x \rightarrow 1 \in E$  and  $(\forall n \in \{2, 3, 4, \dots\}) T^n x - ng = -(n-2) - n(-1) = 2 \in A$  and  $P_A(T^n x - ng) = 2 \in A \setminus E$ . In the proof of Theorem 2.3(i), we had  $(P_A T^n x)_{n \in \mathbb{N}} = (P_A(T^n x - ng))_{n \in \mathbb{N}}$  which is strikingly false here.

### 3 Spingarn's method

In this section we discuss the problem to find *least-squares solutions* of  $\bigcap_{i=1}^M C_i$ , i.e., to

$$\text{find minimizers of } \sum_{i=1}^M d_{C_i}^2, \quad (7)$$

where  $C_1, \dots, C_M$  are nonempty closed convex (possibly nonintersecting) subsets of  $X$  with corresponding distance functions  $d_{C_1}, \dots, d_{C_M}$ . Now consider the product Hilbert space  $\mathbf{X} := X^M$ , with the inner product  $((x_1, \dots, x_M), (y_1, \dots, y_M)) \mapsto \sum_{i=1}^M \langle x_i, y_i \rangle$ . We set

$$\mathbf{A} = \{(x, \dots, x) \in \mathbf{X} \mid x \in X\} \quad \text{and} \quad \mathbf{B} = C_1 \times \dots \times C_M. \quad (8)$$

<sup>3</sup>In [3, Remark 3.14(ii)], the authors claim otherwise but forgot to list the assumption that  $A \cap B \neq \emptyset$ .

Then the projections of  $\mathbf{x} = (x_1, \dots, x_M) \in \mathbf{X}$  onto  $\mathbf{A}$  and  $\mathbf{B}$  are given by, respectively,  $P_{\mathbf{A}}\mathbf{x} = \left(\frac{1}{M} \sum_{i=1}^M x_i, \dots, \frac{1}{M} \sum_{i=1}^M x_i\right)$  and  $P_{\mathbf{B}}\mathbf{x} = (P_{C_1}x_1, \dots, P_{C_M}x_M)$ . Now assume that

$$\mathbf{g} = (g_1, \dots, g_M) := P_{\overline{\mathbf{B}-\mathbf{A}}}0 \in \mathbf{B} - \mathbf{A}. \quad (9)$$

Then we have

$$\mathbf{E} := \mathbf{A} \cap (\mathbf{B} - \mathbf{g}) \neq \emptyset, \quad \text{and} \quad (x, \dots, x) \in \mathbf{A} \cap (\mathbf{B} - \mathbf{g}) \Leftrightarrow x \in \bigcap_{j=1}^M (C_j - g_j). \quad (10)$$

Using [1, Section 6], we see that the  $M$ -set problem (7) is equivalent to the *two-set* problem

$$\text{find least-squares solutions of } \mathbf{A} \cap \mathbf{B}. \quad (11)$$

It follows from (9) and (10) that  $\mathbf{g}$  is the unique vector in  $\overline{\mathbf{B} - \mathbf{A}}$  that satisfies

$$\left. \begin{aligned} (w_1, w_2, \dots, w_M) &\neq (g_1, g_2, \dots, g_M), \\ \text{and } \bigcap_{j=1}^M (C_j - w_j) &\neq \emptyset \end{aligned} \right\} \Rightarrow \sum_{j=1}^M \|w_j\|^2 > \sum_{j=1}^M \|g_j\|^2. \quad (12)$$

We have the following result for the problem of finding a least-squares solution for the intersection of a finite family of sets.

**Corollary 3.1.** *Suppose that  $C_1, \dots, C_M$  are closed convex subsets of  $X$ . Let  $\mathbf{T} = \text{Id} - P_{\mathbf{A}} + P_{\mathbf{B}}R_{\mathbf{A}}$ , let  $\mathbf{x} \in \mathbf{X}$  and recall assumption (9). Then the shadow sequence  $(P_{\mathbf{A}}\mathbf{T}^n\mathbf{x})_{n \in \mathbb{N}}$  converges to  $\bar{\mathbf{x}} = (\bar{x}, \dots, \bar{x}) \in \mathbf{A} \cap (\mathbf{B} - \mathbf{g})$ , where  $\bar{x} \in \bigcap_{j=1}^M (C_j - g_j)$  and  $\bar{x}$  is a least-squares solution of (7).*

*Proof.* Combine Theorem 2.3 with (12) and (10). ■

**Remark 3.2.** When we particularize Corollary 3.1 from convex sets to *halfspaces* and  $X$  is *finite-dimensional*, we recover Spingarn's [11, Theorem 1]. Note that in this case, in view of [1, Facts 5.1(ii)] we have  $\mathbf{g} \in \mathbf{B} - \mathbf{A}$ . Recall that Spingarn used the following version of his *method of partial inverses* from [10]:

$$(\mathbf{a}_0, \mathbf{b}_0) \in \mathbf{A} \times \mathbf{A}^\perp \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \begin{cases} \mathbf{a}'_n = P_{\mathbf{B}}(\mathbf{a}_n + \mathbf{b}_n), & \mathbf{b}'_n = \mathbf{a}_n + \mathbf{b}_n - \mathbf{a}'_n, \\ \mathbf{a}_{n+1} = P_{\mathbf{A}}\mathbf{a}'_n, & \mathbf{b}_{n+1} = \mathbf{b}'_n - P_{\mathbf{A}}\mathbf{b}'_n. \end{cases} \quad (13)$$

This method is the DRA in  $\mathbf{X}$ , applied to  $\mathbf{A}$  and  $\mathbf{B}$  with starting point  $(\mathbf{a}_0 - \mathbf{b}_0)$  (see, e.g., [4, Lemma 2.17]).

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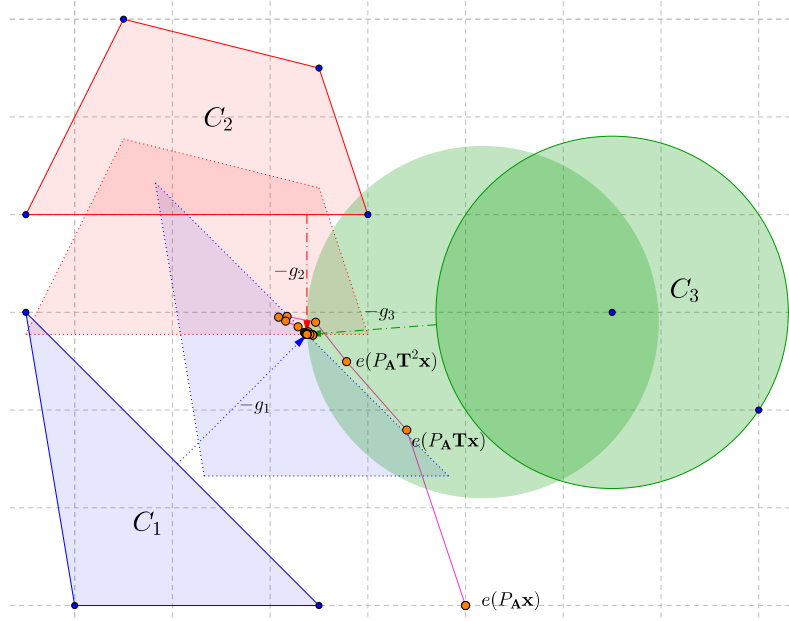


Figure 1: A GeoGebra [7] snapshot that illustrates Corollary 3.1. Three nonintersecting closed convex sets,  $C_1$  (the blue triangle),  $C_2$  (the red polygon) and  $C_3$  (the green circle), are shown along with their translations forming the generalized intersection. The first few terms of the sequence  $(e(P_A T^n \mathbf{x}))_{n \in \mathbb{N}}$  (yellow points) are also depicted. Here  $e : \mathbf{A} \rightarrow \mathbb{R}^2 : (x, x, x) \mapsto x$ .

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